Linear Algebra

Elijah Renner

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1 Basis Vectors

The standard basis vectors in three dimensions and their coordinates are:

$$\hat{\mathbf{i}} = (1, 0, 0), \quad \hat{\mathbf{j}} = (0, 1, 0), \quad \hat{\mathbf{k}} = (0, 0, 1)$$

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This means they can also be expressed as 3D vectors. We say a vector is $n-$dimensional if it has n entries. We can also indicate a vector $\vec{v}$ is $n-$dimensional by saying $\vec{v} \in \mathbb{R}^n$. Here, our basis vectors are in $\mathbb{R}^3$.

Let's expand our definition of standard basis vectors to $\mathbb{R}^n$:

Consider a vector $\vec{v}$ in $\mathbb{R}^n$. The standard basis vectors in $\mathbb{R}^n$ are:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

2 Matrices as Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$[1] be a linear transformation. The matrix $A \in \mathbb{R}^{m\times n}$ representing T can be formed as follows:

- Let $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis vectors in $\mathbb{R}^n$.

$$\vec{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \quad \dots, \quad \vec{e}_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}.$$

- Apply the linear transformation T to each basis vector:

$$T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n).$$

- Form the matrix A by placing the transformed basis vectors as columns:

$$A = \begin{pmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{pmatrix}.$$

- The i-th column of A is $T(\vec{e}_i)$.
- For any vector $\vec{x} \in \mathbb{R}^n$,

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n,$$

  we have:

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n).$$

  This is equivalent to:

$$T(\vec{x}) = A\vec{x}.$$

3 Linear Combinations

Any vector $\vec{v} \in \mathbb{R}^n$ with components $(v_1, v_2, \dots, v_n)$ can be written as:

$$\vec{v} = \begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}.$$

This vector can be expressed as a linear combination of the basis vectors:

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n.$$

For example, in $\mathbb{R}^3$, the vector $\vec{v}$ with components (v_1, v_2, v_3) can be written as:

$$\vec{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1\begin{pmatrix}1\\0\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\\0\end{pmatrix} + v_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

[1] This notation means that T is a mapping from $\mathbb{R}$ to $\mathbb{R}$

4 Span of Vectors

The span of a set of vectors is the set of all possible linear combinations of those vectors. If you have a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, the span of these vectors is denoted as $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ and is defined as:

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k \mid a_1, a_2, \ldots, a_k \in \mathbb{R}\}.$$

This set includes all vectors that can be formed by taking linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

Note: if two vectors $\vec{v}$ and $\vec{k}$ are colinear, their span is a line.

5 Colinearity

A vector $\vec{v}$ is colinear with $\vec{k}$ if $\vec{v} = a\vec{k}$ for some scalar a.

6 Linear Independence and Dependence

6.1 Linear Independence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is said to be linearly independent if no vector in the set can be written as a linear combination of the others. Formally, the vectors are linearly independent if the only solution to the equation

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k = \vec{0}$$

is $a_1 = a_2 = \cdots = a_k = 0$. This means that the only way to get the zero vector using a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is by setting all the coefficients to zero.

6.2 Linear Dependence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is said to be linearly dependent if at least one vector in the set can be written as a linear combination of the others. Formally, the vectors are linearly dependent if there exists a non-trivial solution (that is, a nonzero one) to the equation

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}.$$

This means that there are some non-zero coefficients that can be used to express the zero vector as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

7 Row Operations of Matrices

Row operations are used to manipulate matrices, especially for solving linear systems and performing Gaussian elimination. There are three types of row operations:

1. **Row Switching:** Swap the positions of two rows. Symbol: $\iff$
   Example: Switch row i with row j.
   $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{\text{switch } R_1 \text{ and } R_2} \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$
2. **Row Multiplication:** Multiply all elements of a row by a nonzero scalar.
   Example: Multiply row i by $c \neq 0$.
   $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{\text{multiply } R_2 \text{ by } 2} \begin{pmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{pmatrix}$$

3. **Row Addition:** Add or subtract the elements of one row to/from another row.
   Example: Add c times row i to row j.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{\text{add } 2R_1 \text{ to } R_2} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 9 \end{pmatrix}$$

8 Matrix Multiplication

When multiplying matrix B by matrix A (AB), A must have the same number of columns as the number of rows in B.