MATH1034OL1 Pre-Calculus Mathematics Notes from Sections 4.7, 3.2 (Monday)

Elijah Renner

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Contents

1 Identities

1.1 Pythagorean Identities

We first derive the identity $\cos^2\theta + \sin^2\theta = 1$ from the unit circle.

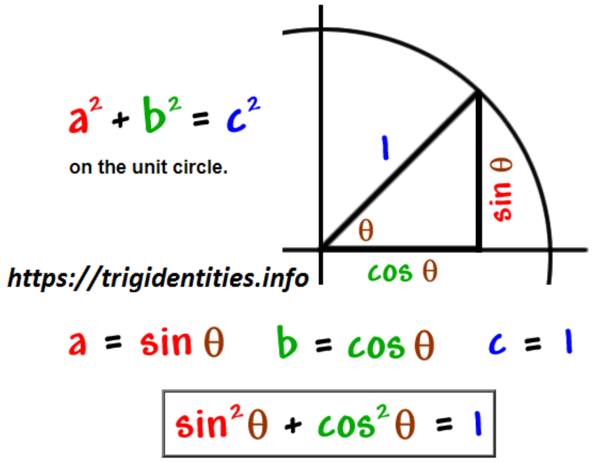

Figure 1: Credit: https://trigidentities.info/pythagorean-trig-identities/

Then, the other two Pythagorean identities are derived by dividing by either $\cos^2$ or $\sin^2$:

To derive $\tan^2\theta + 1 = \sec^2\theta$, divide the original identity by $\cos^2\theta$:

$$\frac{\cos^2\theta + \sin^2\theta = 1}{\cos^2\theta} \implies \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \tan^2\theta + 1 = \sec^2\theta$$

To derive $\cot^2\theta + 1 = \csc^2\theta$, divide the original identity by $\sin^2\theta$:

$$\frac{\cos^2\theta + \sin^2\theta = 1}{\sin^2\theta} \implies \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \implies \cot^2\theta + 1 = \csc^2\theta$$

To summarize, the three Pythagorean identities are

1. $\cos^2\theta + \sin^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $\cot^2\theta + 1 = \csc^2\theta$

1.2 Sum and Difference Formulas

The sum and difference formulas allow us to evaluate the trigonometric functions of angles whos reference angles aren't 30, 45, 60, or 90:

$$\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$$

$$\cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B$$

There is a formula for $\tan(A \pm B)$, which isn't necessary to remember, as it can be derived from $\frac{\sin(A \pm B)}{\cos(A \pm B)}$ since $\tan\theta = \frac{\sin\theta}{\cos\theta}$. The same follows for $\csc(A \pm B) = \frac{1}{\sin(A \pm B)}$, $\sec(A \pm B) = \frac{1}{\csc(A \pm B)}$, and $\cot(A \pm B) = \frac{\cos(A \pm B)}{\sin(A \pm B)}$.

Regardless,

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$$

.
Also, $\mp$ indicates the opposite sign of whichever sign is chosen as $\pm$.

1.3 Double Angle Formulas

To derive the double angle formulas, we start with the angle familiar sum identities.

1.3.1 Sine

The angle sum identity for sine is:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

By setting $A = B = \theta$, we get:

$$\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

Simplify this by combining like terms:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

1.3.2 Cosine

The angle sum identity for cosine is:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

By setting $A = B = \theta$, we get:

$$\cos(2\theta) = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

Simplify this by combining like terms:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can derive alternative forms of $\cos(2\theta)$:

1. Express $\cos^2\theta$ in terms of $\sin^2\theta$:

$$\cos^2\theta = 1 - \sin^2\theta$$

Substitute this into the double angle formula for cosine:

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

2. Express $\sin^2\theta$ in terms of $\cos^2\theta$:

$$\sin^2\theta = 1 - \cos^2\theta$$

Substitute this into the double angle formula for cosine:

$$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

So, we have three equivalent forms of $\cos(2\theta)$:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

These derivations give us the double angle formulas for sine and cosine:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1 \end{aligned}$$

Nice!

1.4 Half Angle Formulas

The half angle formulas for sin and cos are

$$\begin{aligned} \sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) &\implies \sin\theta = \pm\sqrt{\frac{1}{2}(1 - \cos(2\theta))} = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}} \\ \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) &\implies \cos\theta = \pm\sqrt{\frac{1}{2}(1 + \cos(2\theta))} = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}} \end{aligned}$$

To derive the half-angle formula for tan, we use the half-angle formulas for sin and cos:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Using the half-angle formulas:

$$\sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

So,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\pm\sqrt{\frac{1-\cos(2\theta)}{2}}}{\pm\sqrt{\frac{1+\cos(2\theta)}{2}}}$$

Since both the numerator and the denominator have the same sign, the signs cancel out:

$$\tan\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} \div \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

Thus, the half-angle formula for tan is:

$$\tan\theta = \sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

1.5 Quizlet

I know that was a lot. There's a lot to remember, so I'll be quizzing myself. Here is my quizlet:

Quizlet Link

You might bookmark this, since I'll be updating it as we learn more identities.

2 Vertex of Quadratic

Let $f(x) = ax^2 + bx + c$ where a, b, and c are constants. The vertex of f will always be

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

3 Polynomial Behavior

If a factor $(x - a)$ appears an even amount of times, the function will touch the x-axis when $x = a$.

Conversely, if $(x - a)$ appears an odd amount of times, the function will cross the x-axis when $x = a$:

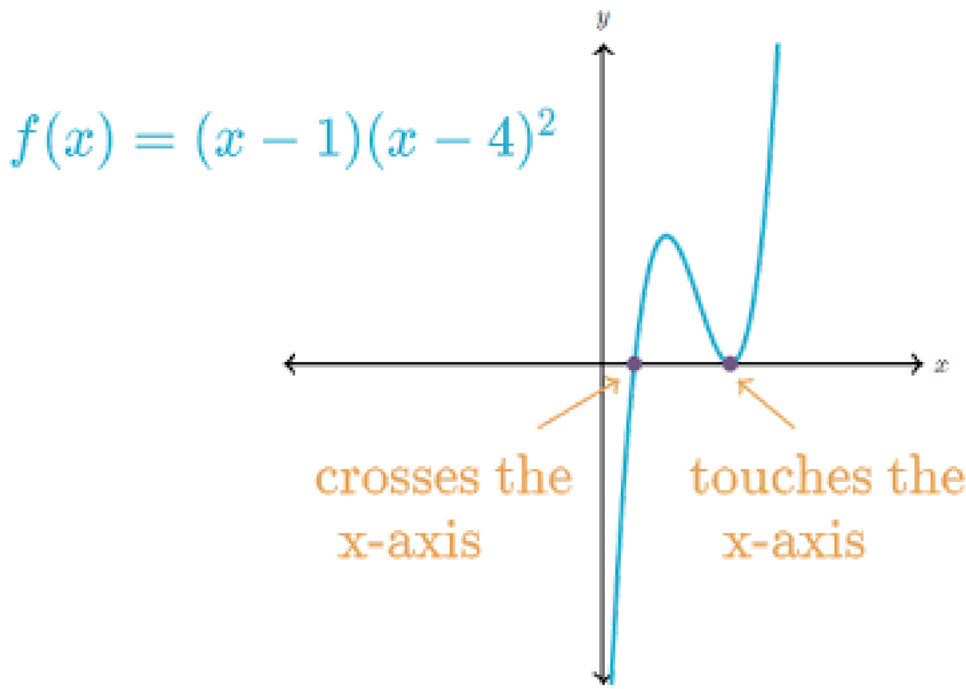

Figure 2: Credit: https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:poly-graphs/x2ec2f6f830c9fb89:poly-intervals/a/zeros-of-polynomials-and-their-graphs

In class, we reviewed the end behaviors of polynomials. I've already recorded them in the notes from July 5th. Good luck on Friday!