MATH1034OL1 Pre-Calculus Mathematics Midterm Review (Wednesday)

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Contents

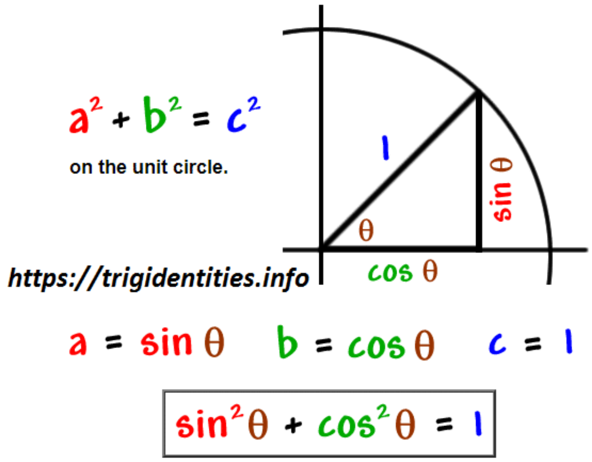
1 About Today's Notes

Since we didn't cover any new content Wednesday, I'm compiling all key information for the midterm here.

2 Identities

2.1 Pythagorean Identities

We first derive the identity $\cos^2\theta + \sin^2\theta = 1$ from the unit circle.



Identity.png

Figure 1: Credit: https://trigidentities.info/pythagorean-trig-identities

Then, the other two Pythagorean identities are derived by dividing by either $\cos^2$ or $\sin^2$:

To derive $\tan^2\theta + 1 = \sec^2\theta$, divide the original identity by $\cos^2\theta$:

$$\frac{\cos^2\theta + \sin^2\theta = 1}{\cos^2\theta} \implies \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \tan^2\theta + 1 = \sec^2\theta$$

To derive $\cot^2\theta + 1 = \csc^2\theta$, divide the original identity by $\sin^2\theta$:

$$\frac{\cos^2\theta + \sin^2\theta = 1}{\sin^2\theta} \implies \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \implies \cot^2\theta + 1 = \csc^2\theta$$

To summarize, the three Pythagorean identities are

1. $\cos^2\theta + \sin^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $\cot^2\theta + 1 = \csc^2\theta$

2.2 Sum and Difference Formulas

The sum and difference formulas allow us to evaluate the trigonometric functions of angles whos reference angles aren't 30, 45, 60, or 90:

$$\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$$

$$\cos(A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B$$

There is a formula for $\tan(A \pm B)$, which isn't necessary to remember, as it can be derived from $\frac{\sin(A \pm B)}{\cos(A \pm B)}$ since $\tan\theta = \frac{\sin\theta}{\cos\theta}$. The same follows for $\csc(A \pm B) = \frac{1}{\sin(A \pm B)}$, $\sec(A \pm B) = \frac{1}{\csc(A \pm B)}$, and $\cot(A \pm B) = \frac{\cos(A \pm B)}{\sin(A \pm B)}$.

Regardless,

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$$

.
Also, $\mp$ indicates the opposite sign of whichever sign is chosen as $\pm$.

2.3 Double Angle Formulas

To derive the double angle formulas, we start with the angle familiar sum identities.

2.3.1 Sine

The angle sum identity for sine is:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

By setting $A = B = \theta$, we get:

$$\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

Simplify this by combining like terms:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

2.3.2 Cosine

The angle sum identity for cosine is:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

By setting $A = B = \theta$, we get:

$$\cos(2\theta) = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

Simplify this by combining like terms:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can derive alternative forms of $\cos(2\theta)$:

1. Express $\cos^2\theta$ in terms of $\sin^2\theta$:

$$\cos^2\theta = 1 - \sin^2\theta$$

Substitute this into the double angle formula for cosine:

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

2. Express $\sin^2\theta$ in terms of $\cos^2\theta$:

$$\sin^2\theta = 1 - \cos^2\theta$$

Substitute this into the double angle formula for cosine:

$$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

So, we have three equivalent forms of $\cos(2\theta)$:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

These derivations give us the double angle formulas for sine and cosine:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned}$$

Nice!

2.4 Half Angle Formulas

The half angle formulas for sin and cos are

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) \implies \sin\theta = \pm\sqrt{\frac{1}{2}(1 - \cos(2\theta))} = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) \implies \cos\theta = \pm\sqrt{\frac{1}{2}(1 + \cos(2\theta))} = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

To derive the half-angle formula for tan, we use the half-angle formulas for sin and cos:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Using the half-angle formulas:

$$\sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$

$$\cos\theta = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

So,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\pm\sqrt{\frac{1-\cos(2\theta)}{2}}}{\pm\sqrt{\frac{1+\cos(2\theta)}{2}}}$$

Since both the numerator and the denominator have the same sign, the signs cancel out:

$$\tan\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} \div \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

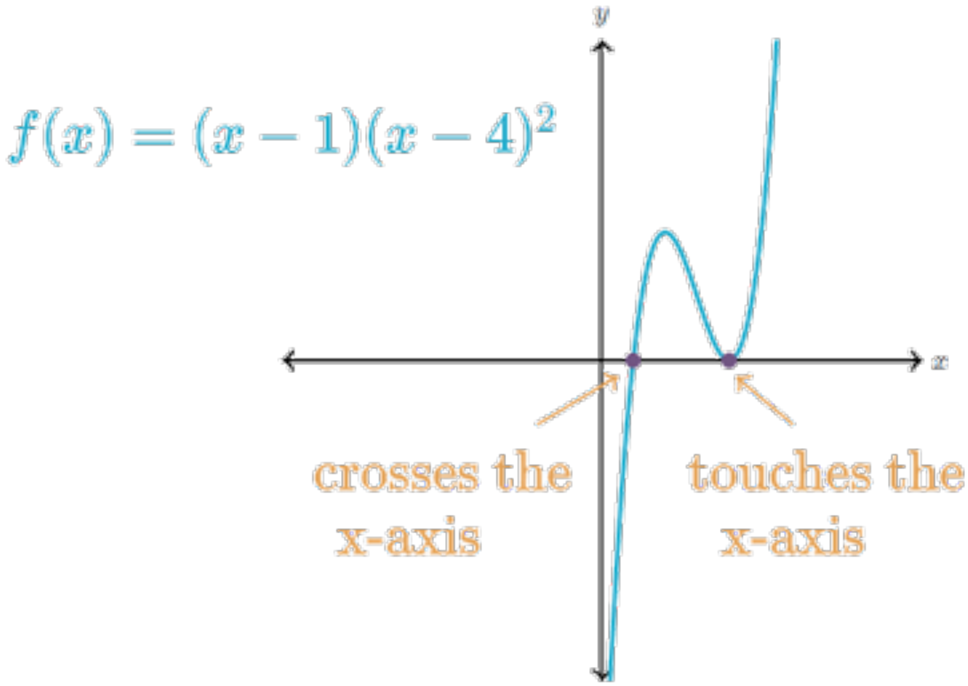
Thus, the half-angle formula for tan is:

$$\tan\theta = \sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$$

3 Polynomial Behavior

If a factor $(x-a)$ appears an even amount of times, the function will touch the x-axis when $x=a$.

Conversely, if $(x-a)$ appears an odd amount of times, the function will cross the x-axis when $x=a$:



touching.png

Figure 2: Credit: https://www.khanacademy.org/math/algebra2/x2ec2f6f830c9fb89:poly-graphs/x2ec2f6f830c9fb89:poly-intervals/a/zeros-of-polynomials-and-their-graphs

4 Quadrant of Manipulated Angle

Problem: if angle θ is in quadrant four, which quadrant is $\frac{\theta}{2}$ in?

We know that a point on an axis is not in a quadrant, so the range of θ is $270° < \theta < 360°$ or $\frac{3\pi}{2} < \theta < 2\pi$. To find the range of $\frac{\theta}{2}$, treat $\frac{3\pi}{2} < \theta < 2\pi$ as an inequality.

Recall that we can divide each part of an inequality by a non-negative number without changing its sides. So, we'll divide by 2 to get $\frac{\theta}{2}$ in the center:

$$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$$

Since this range is in quadrant two, we know $\frac{\theta}{2}$ must be in quadrant two.

5 Absolute Value Equations

Value Equations Image.gif

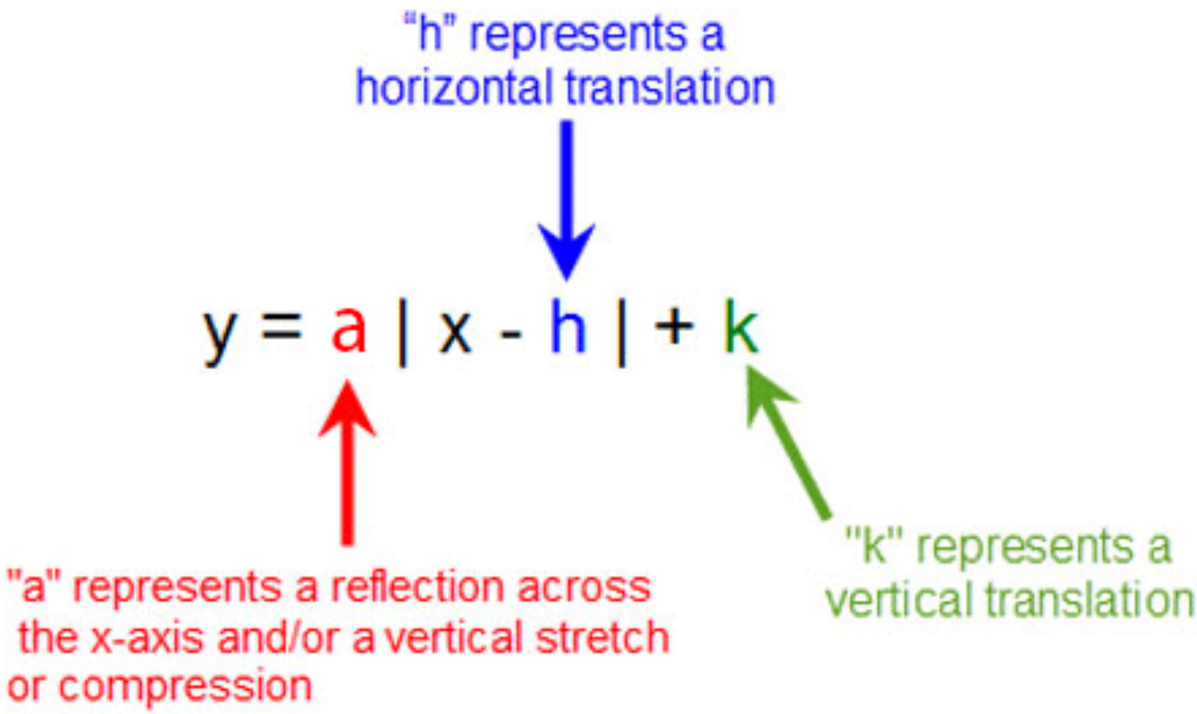


Figure 3: Credit: https://superbiamk.shop/product_details/85110602.html

6 Root Equations

Functions Screenshot Aug 26.png

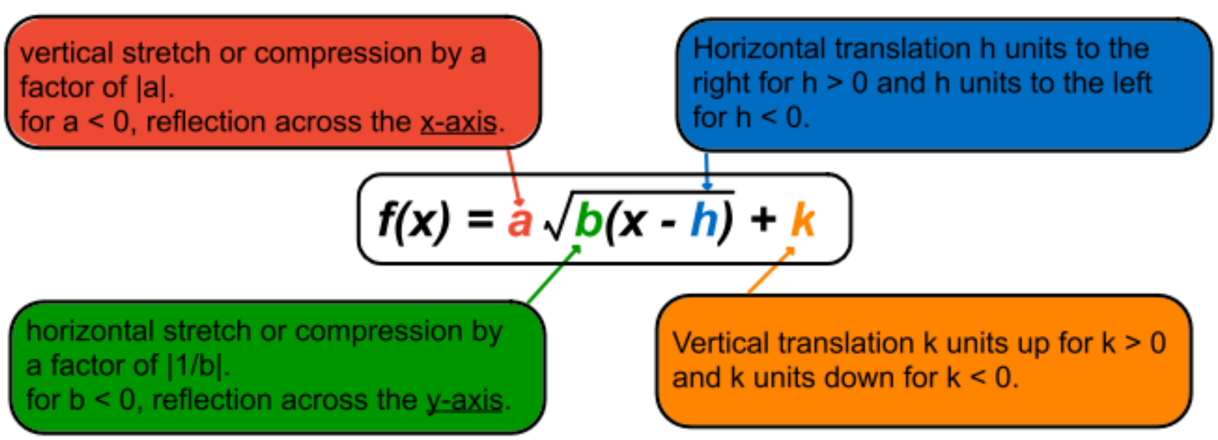


Figure 4: Credit: https://amandapaffrath.weebly.com/square-root-functions.html

7 Sinusoidal Functions

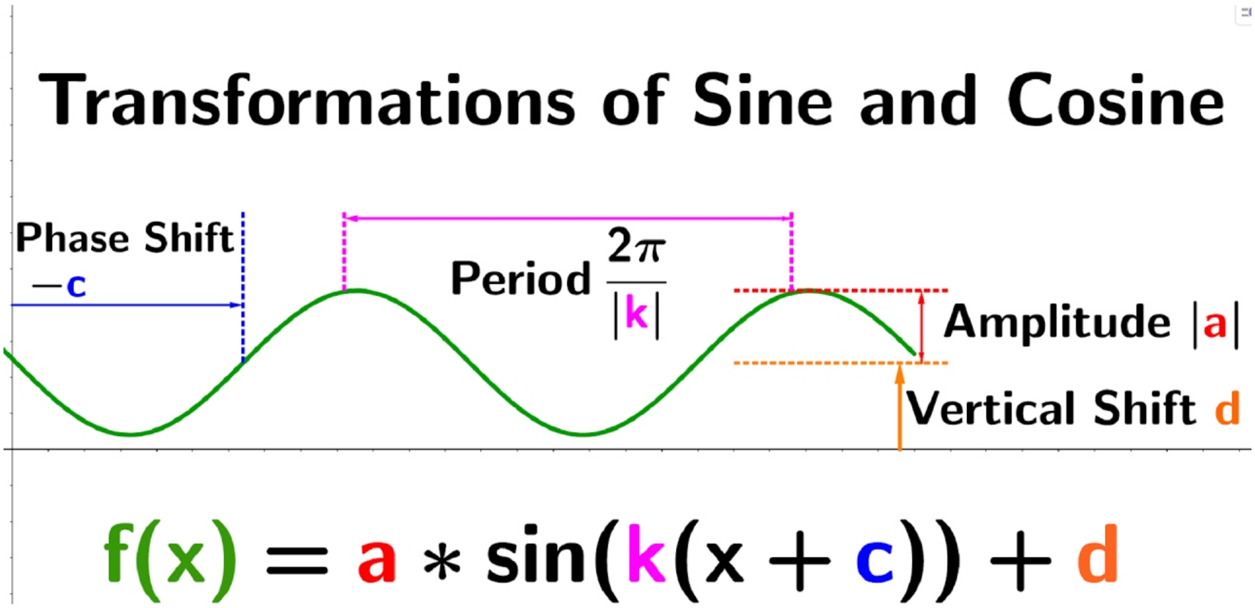


Figure 5: Credit: https://www.youtube.com/watch?v=AS7THLj-OhI

8 Domain

8.1 Domain of Root Functions

The domain of n-th root functions depends on whether n is even or odd:

1. **For even n (e.g., square root, fourth root, etc.):**

   The n-th root function is defined for all non-negative values of the radicand. This is because taking an even root of a negative number is not defined in the real number system.

$$f(x) = \sqrt[n]{x}$$

   Domain: $x \geq 0$

2. **For odd n (e.g., cube root, fifth root, etc.):**

   The n-th root function is defined for all real numbers. This is because taking an odd root of a negative number is defined and results in a negative number.

$$f(x) = \sqrt[n]{x}$$

   Domain: $x \in \mathbb{R}$ (all real numbers)

Figure 6: Credit: myself

8.2 Domain of Polynomials

All n-th degree polynomials in the form $a_n x^n + a_{n-1} x^{n-1} ... + a_2 x^2 + a_1 x + a_0$ are defined for all $x \in \mathbb{R}$.

8.3 Domain of Rational Functions

Rational functions $\frac{f(x)}{p(x)}$ are defined for all values where $p(x) \neq 0$.

8.4 Domain of Added, Subtracted, Multiplied, and Divided Functions

1. **Addition and Subtraction ($f(x) + g(x)$ and $f(x) - g(x)$):**
   - The domain of $f + g$ (or $f - g$) is the intersection of the domains of f and g.
   - Mathematically: $\text{Domain}(f+g) = \text{Domain}(f) \cap \text{Domain}(g)$.
2. **Multiplication ($f(x) \cdot g(x)$):**
   - The domain of $f \cdot g$ is also the intersection of the domains of f and g.
   - Mathematically: $\text{Domain}(f \cdot g) = \text{Domain}(f) \cap \text{Domain}(g)$.
3. **Division ($\frac{f(x)}{g(x)}$):**
   - The domain of $\frac{f}{g}$ is the intersection of the domains of f and g, excluding the points where $g(x) = 0$ (since division by zero is undefined).
   - Mathematically: $\text{Domain}\left(\frac{f}{g}\right) = \text{Domain}(f) \cap \text{Domain}(g) - \{x \mid g(x) = 0\}$.

Figure 7: Credit: myself

8.5 Domain of Composed Functions

Here's a review of set builder notation:

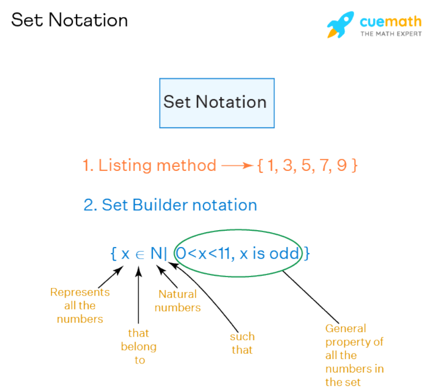


Figure 8: Credit: https://www.cuemath.com/algebra/set-builder-notation/

$$\text{Domain}(g \circ f) = \{x \in \text{Domain}(g) \mid g(x) \in \text{Domain}(f)\}$$

9 Limit Definition of the Derivative

The derivative (or instantaneous rate of change) of a function f at a point $x = a$ is defined by the limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

10 Rationalizing Numerator and Denominator

Conjugates are helpful for rationalizing numerators and denominators:

2. Rationalizing the Denominator

When dealing with expressions that have radicals in the denominator, multiplying by the conjugate can rationalize the denominator.

Example:

$\frac{1}{\sqrt{3}+1}$

Multiply the numerator and the denominator by the conjugate of the denominator:

$\frac{1}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{\sqrt{3}-1}{(\sqrt{3})^2-1^2} = \frac{\sqrt{3}-1}{3-1} = \frac{\sqrt{3}-1}{2}$

Figure 9: Credit: myself

11 Inverse Functions

11.1 Finding Inverses

Given a function $f(x)$, we denote its inverse as $f^{-1}(x)$. To find the inverse of $f(x)$:

1. Replace $f(x)$ with y.
2. Switch y and x.
3. Solve for y.
4. Replace y with $f^{-1}(x)$

Here's an example:

Problem: Let $f(x) = x^2 + 3$. Find $f^{-1}(x)$.

Replace $f(x)$ with y: $f(x) = x^2 + 3 \implies y = x^2 + 3$

Switch x and y: $y = x^2 + 3 \implies x = y^2 + 3$

Solve for y: $x = y^2 + 3 \implies y^2 = x - 3 \implies y = \sqrt{x - 3}$

Replace y with $f^{-1}(x)$: $f^{-1}(x) = \sqrt{x - 3}$

Now we know $f^{-1}(x) = \sqrt{x - 3}$.

11.2 Properties of Inverses

A cool property of inverse functions is that $f(x)$ and $f^{-1}(x)$ reflect over the line $y = x$:

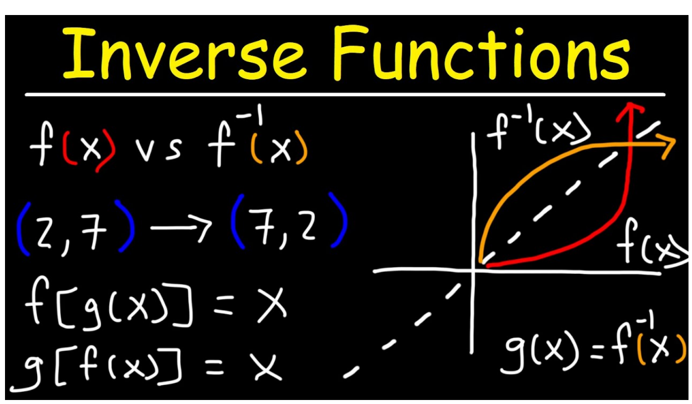


Figure 10: Credit: https://youtu.be/TN4ybFiuV3k

This makes sense because we are switching x and y.

Another property of f^{-1} is that its domain and range are the range and domain respectfully of f. More formally:

domain(f) = range(f^{-1})

range(f) = domain(f^{-1})

12 Review of Linear Functions

To find the slope of a line given two points, we use *slope* $= m = \frac{\Delta y}{\Delta x}$. Given two points (x_1, y_1) and (x_2, y_2), we can calculate the slope using

$$\frac{y_1 - y_2}{x_1 - x_2}$$

If we a line's slope m and a point (x_1, y_1) on the line, we can write its equation as

$$y - y_1 = m(x - x_1)$$

Some rules:
A line parallel to a line with slope m will have slope m
A line perpendicular to a line with slope m will have slope $-\frac{1}{m}$
A line written in the form $Ax + By = C$ will have slope $m = -\frac{A}{B}$ and y-intercept $b = \frac{C}{B}$

13 End Behaviors

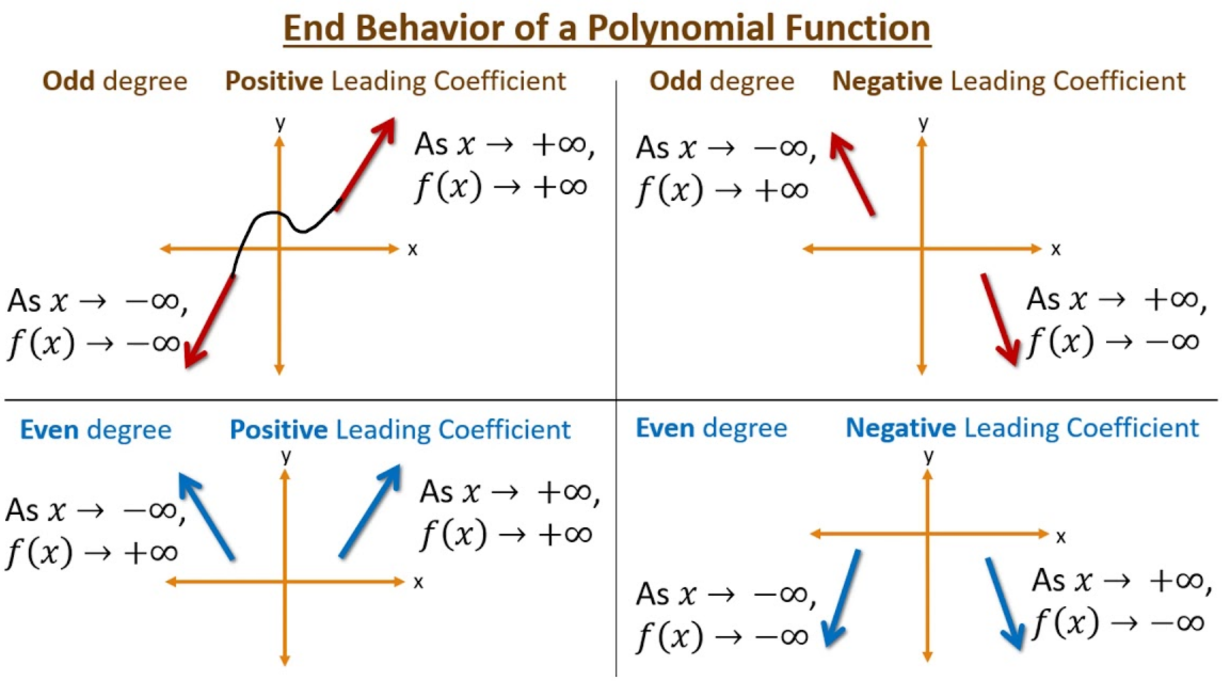


Figure 11: Credit: https://youtu.be/7LnsYtCfkXQ?si=Iq8WqdaHbLvWcLC0

14 Transformations of Functions

Here's a helpful graphic about transforming functions:

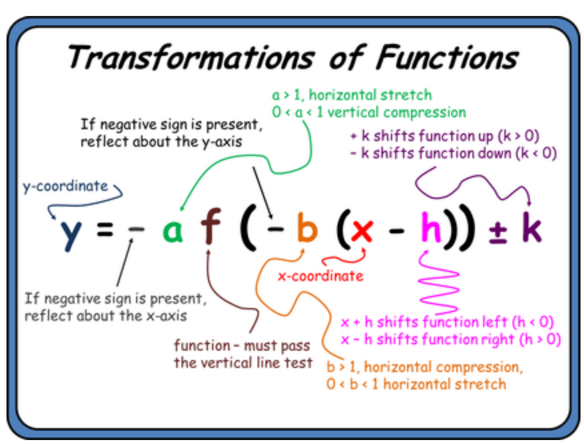


Figure 12: Credit:
https://lzinnick.weebly.com/transformations-of-functions-and-graphs.html

15 Special Triangles

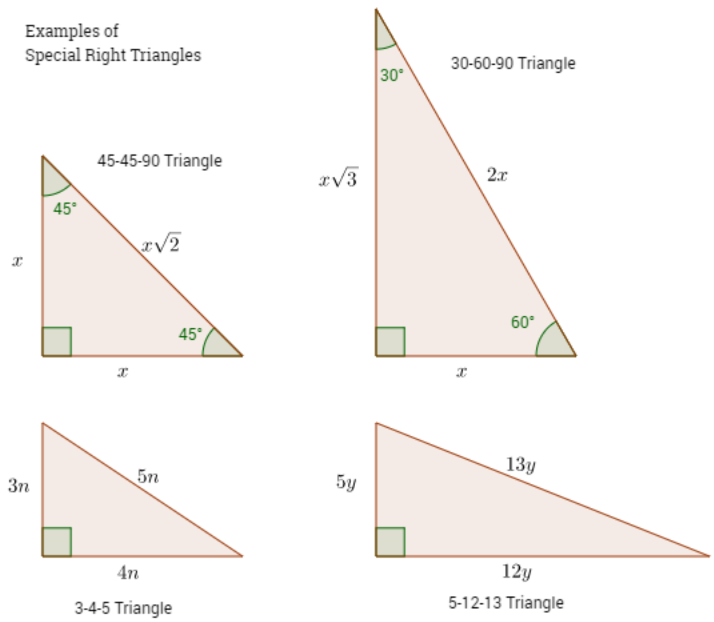


Figure 13: Credit: https://www.onlinemathlearning.com/special-right-triangles.html

16 All Students Take Calculus

"All Students Take Calculus." tells us which trig functions are positive in each quadrant.

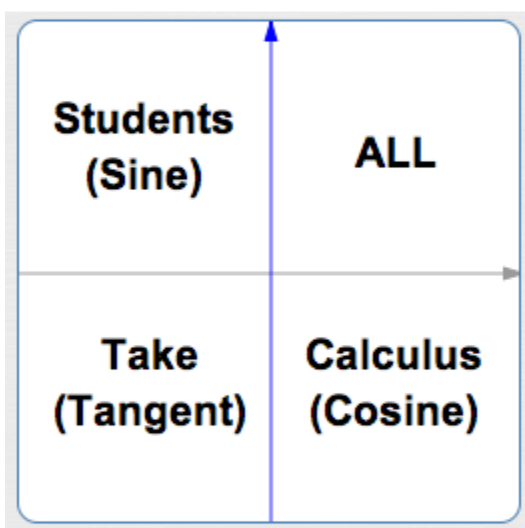


Figure 14: Credit: https://www.onemathematicalcat.org

17 Finding the Radius and Center of Circle by Completing the Square

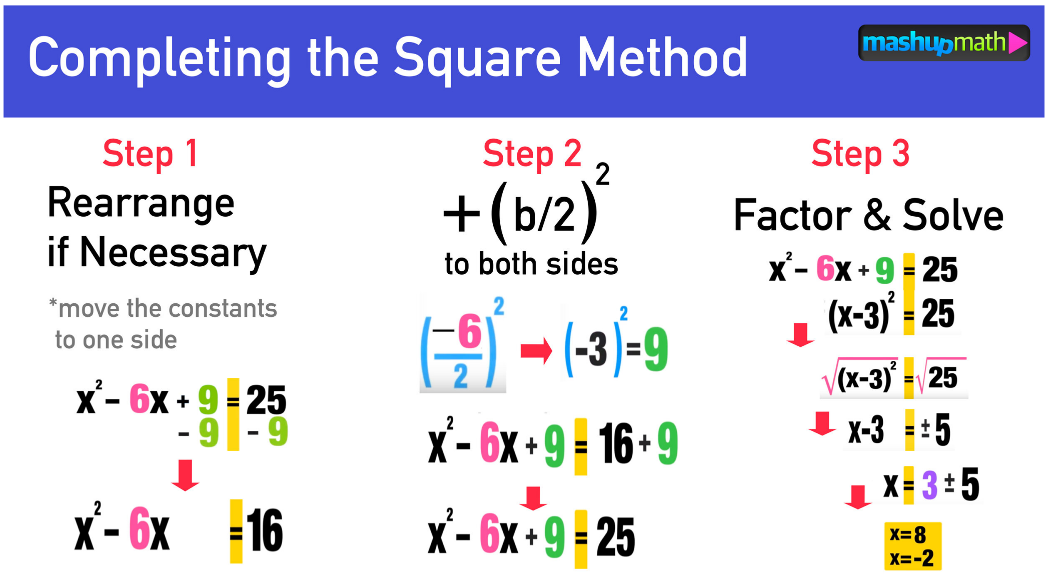


Figure 15: Credit: https://www.mashupmath.com/blog/complete-the-square-formula

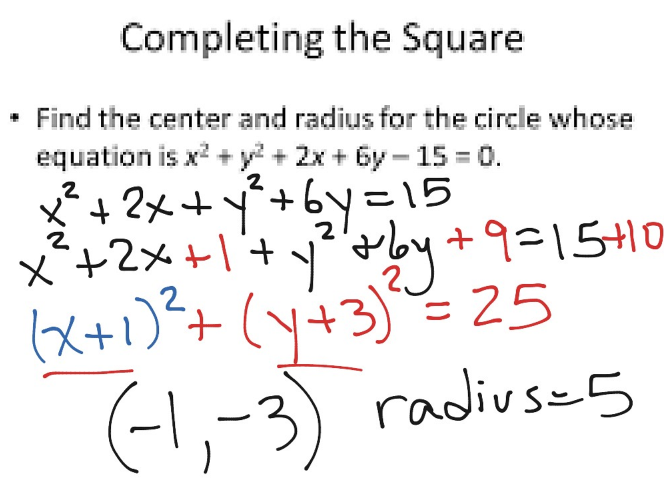


Figure 16: Credit: https://www.showme.com/sh/?h=nFTWVUG

18 Converting Between Radians and Degrees

Here are the methods for converting between radians and degrees:

$$\text{Radians} = \text{Degrees} \times \frac{\pi}{180}$$

$$\text{Degrees} = \text{Radians} \times \frac{180}{\pi}$$

19 Reference Angles

We define reference angles as the smallest, positive, acute angle formed by the terminal side of an angle and the x-axis on a coordinate plane:

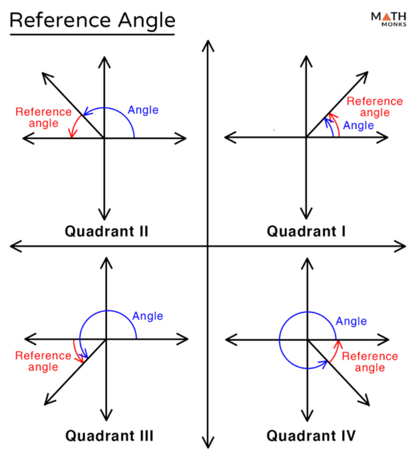


Figure 17: Credit: https://mathmonks.com/angle/reference-angle

20 Vertex of Quadratic

Let $f(x) = ax^2 + bx + c$ where a, b, and c are constants. The vertex of f will always be

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

21 Factoring Higher-Degree Polynomials

Let's suppose we have the equation $8x^3 + x + 9$=0. The left side isn't easily factorable and can't be plugged into the quadratic formula. Instead, we can find the possible rational zeros of the expression.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$ where a are integers. Here,

$$\text{possible rational zeros} = \frac{\text{factors of } a_0 \text{ (last term)}}{\text{factors of } a_n \text{(first term)}}$$

Hence, the possible rational zeroes of $8x^3 + x + 9$ are $\pm\frac{1,3,9}{1,2,4,8} = \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{3}{8}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{9}{8}, \pm\frac{3}{2}, \pm\frac{9}{4}, \pm\frac{9}{2}, \pm 3, \pm 9$.

Personally, I like to start with integers. We can manually plug possible rational zeros in as x. However, there's a trick to quickly evaluate polynomials at a certain value x. This isn't easily explained in writing, so here's a helpful video from the Organic Chemistry tutor.

Just remember that if $f(a) = 0$ than $(x - a)$ is a factor of f.

22 Intersection and Union

The union is the combination of two sets, while the intersection is the overlap between two sets.

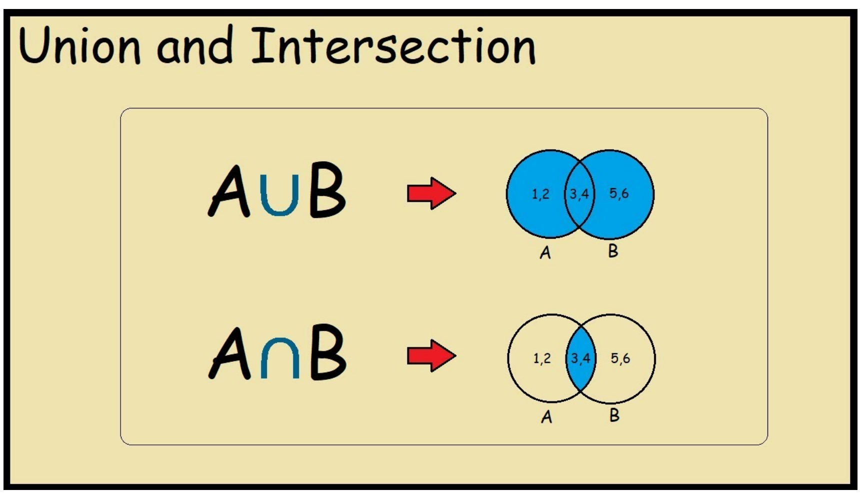


Figure 18: Credit: https://www.youtube.com/watch?v=sdflTUW6gHo