MATH1034OL1 Pre-Calculus Mathematics Notes from Sections 4.5, 1.6, 2.3 (Monday)

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Contents

1 Review Problems

1.1 Problem: Evaluate $\cot\left(\frac{2\pi}{3}\right)$

First, we determine the reference angle. Since $\frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = 120^\circ$, the angle is in the second quadrant. Recall from July 5th's notes that to find the reference angle of an angle in the second quadrant, we subtract it from 180° or π. Since we're working toward being comfortable with radian measures, we'll perform this operation in radians:

$$\text{reference angle} = \pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{\pi}{3}$$

This tells us we're dealing with a 30-60-90 triangle. Now, let's recall that $\cot = \frac{\text{adj}}{\text{opp}}$. Let's look for those.

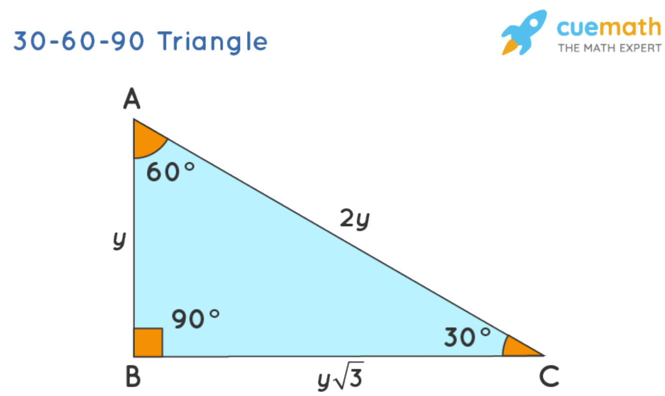

Figure 1: Credit: https://www.cuemath.com/geometry/30-60-90-triangle

Here, $y = \frac{1}{2}$. So, opp $= \frac{\sqrt{3}}{2}$ and adj $= \frac{1}{2}$. Thus, $\cot\left(\frac{2\pi}{3}\right) = \frac{\text{adj}}{\text{opp}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Don't forget that $\cot\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{3}$ because cot is negative in quadrant two.

1.2 Graph $y = 2\sin(4x - 20°) - 3$

When graphing sinosodal functions, we must determine amplitude, period, sub-period, phase (horizontal) shift, vertical shift, and shape:

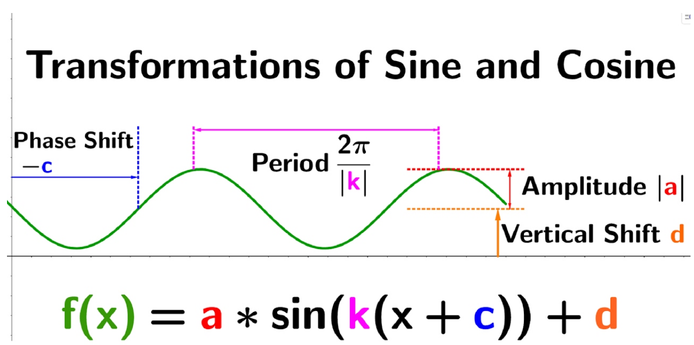

Figure 2: Credit: https://youtu.be/AS7THLj-OhI

We first factor the $4x - 20°$: $y = 2\sin(4x - 20°) - 3 = 2\sin(4(x - 5°)) - 3$.
Now, $a = 2, k = 4, c = -5$, and $d = -3$.

Thus, amplitude $= |a| = |2| = 2$, period $= \frac{2\pi}{|k|} = \frac{2\pi}{4} = \frac{\pi}{2}$, sub-period $= \frac{\text{period}}{4} = \frac{\pi}{8}$, phase shift $= -c = 5$, vertical shift $= d = -3$.

Since $a > 0$, the graph opens up. With these properties, the function can be graphed.

2 Trigonometric Functions on the Unit Circle

On the unit circle with angle θ, $x = \cos\theta$ and $y = \sin\theta$:

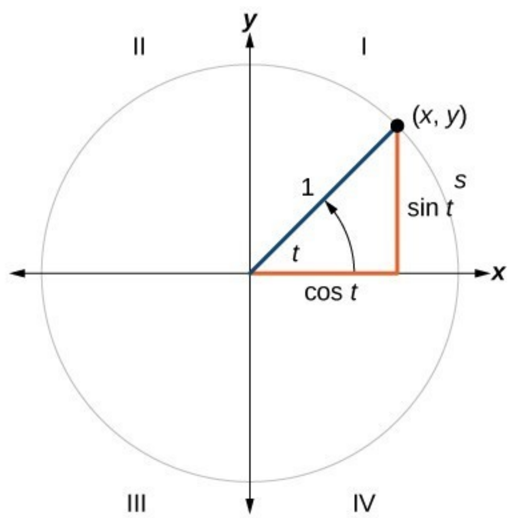

Figure 3: Credit: https://courses.lumenlearning.com/precalculus/chapter/unit-circle-sine-and-cosine-functions

Hence,

$$\begin{aligned}
\tan\theta &= \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \quad (x \neq 0)\\
\csc\theta &= \frac{1}{y} \quad (y \neq 0)\\
\sec\theta &= \frac{1}{x} \quad (x \neq 0)\\
\cot\theta &= \frac{x}{y} = \frac{\cos\theta}{\sin\theta} \quad (y \neq 0)
\end{aligned}$$

3 Domain and Range

Polynomials all have domain $\mathbb{R}$.

For even degree polynomials, the range is restricted.

For odd degree polynomials, the range is $\mathbb{R}$.

Rational functions have a domain of $\mathbb{R}$ excluding the values that make the denominator equal to zero. Example:

for $g(x) = \frac{x-2}{5x+15}$, $5x + 15 \neq 0$.

$\implies 5x \neq -15$

$\implies x \neq -3$

Hence, the domain of g can be written as $\{x \in \mathbb{R} \mid x \neq -3\}$.

Even root functions, such as $h(x) = \sqrt{4x+1}$, have domains of all values x such that the contents of the root aren't negative. For h, $4x + 1 \geq 0$.

$\implies 4x \geq -1$

$\implies x \geq -\frac{1}{4}$

So the domain of h is $[-\frac{1}{4}, \infty)$.

Conversely, odd root functions have domains $\mathbb{R}$.

4 Limit Definition of the Derivative

The derivative of a function $f(x)$ at a point $x = a$ is defined by the limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let's apply this definition to the function $f(x) = x^2 - x + 2$.
First, we need to compute $f(a+h)$:

$$f(a+h) = (a+h)^2 - (a+h) + 2$$

Expanding this, we get:

$$f(a+h) = a^2 + 2ah + h^2 - a - h + 2$$

Next, we compute $f(a)$:

$$f(a) = a^2 - a + 2$$

Now, we find the difference $f(a+h) - f(a)$:

$$f(a+h) - f(a) = (a^2 + 2ah + h^2 - a - h + 2) - (a^2 - a + 2)$$

Simplifying, we get:

$$f(a+h) - f(a) = 2ah + h^2 - h$$

We then divide by h:

$$\frac{f(a+h) - f(a)}{h} = \frac{2ah + h^2 - h}{h} = 2a + h - 1$$

Finally, we take the limit as h approaches 0:

$$f'(a) = \lim_{h \to 0}(2a + h - 1) = 2a - 1$$

Therefore, the derivative of the function $f(x) = x^2 - x + 2$ is:

$$f'(x) = 2x - 1$$

To find the slope of the function at $x = 5$, we substitute $x = 5$ into the derivative:

$$f'(5) = 2(5) - 1$$

Simplifying, we get:

$$f'(5) = 10 - 1 = 9$$

Therefore, the slope of the function $f(x) = x^2 - x + 2$ at the point $x = 5$ is:

$$f'(5) = 9$$

Now, let's use the point-slope form to find the equation of the tangent line at $x = 5$.
The point-slope form of the equation of a line is given by:

$$y - y_1 = m(x - x_1)$$

where m is the slope of the line and (x_1, y_1) is a point on the line.
In our case, the slope m is 9, and the point (x_1, y_1) is $(5, f(5))$.
First, we need to find $f(5)$:

$$f(5) = 5^2 - 5 + 2 = 25 - 5 + 2 = 22$$

So the point is $(5, 22)$.
Now, we can substitute $m = 9$, $x_1 = 5$, and $y_1 = 22$ into the point-slope form:

$$y - 22 = 9(x - 5)$$

Simplifying, we get:

$$y - 22 = 9x - 45$$

$$y = 9x - 45 + 22$$

$$y = 9x - 23$$

Therefore, the equation of the tangent line to the function $f(x) = x^2 - x + 2$ at the point $x = 5$ is:

$$y = 9x - 23$$