MATH1034OL1 Pre-Calculus Mathematics Sections Final Review and Cheatsheet for Calculus

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Contents

1 About

Here's all the information you need for the midterm.

2 Content Before Midterm

In order to avoid redundancy, only content from after the midterm will be included. See the the midterm review for content before the review: https://tamath.org/media/notebooks/precalculus/7.17/7.17.pdf.

3 Inverse Trigonometric Functions

The three inverse trigonometric functions are arccos, arcsin, and arctan. These functions have the property

$$\arccos(x) = A$$
$$\cos(A) = x$$

In other words, cos and arccos are inverses of each other. They're helpful for determining an angle given its $\sin, \cos$, or tan value.

When we're finding $\arcsin(x)$, we restrict the range of arcsin such that it remains a function (that is, there are no two outputs $\arcsin(x)$ for one input x). We do this for all inverse trigonometric functions. Here are the restrictions:

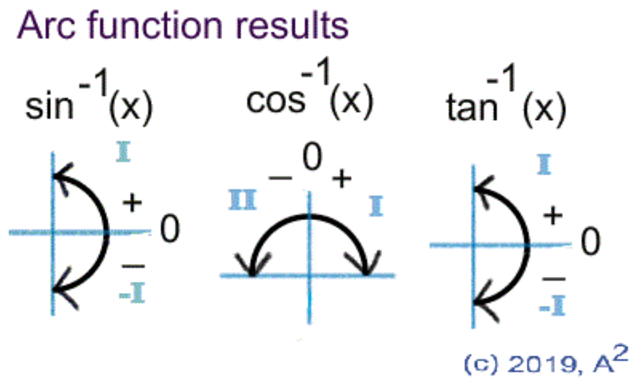

Figure 1: Credit: https://www.mathnstuff.com/math/spoken/here/2class/330/arc.htm

Problem: Find the Exact value of $\sec\{\arctan[\sin(\arccos(\frac{-1}{2}))]\}$.

First, recall that $\arccos(\frac{-1}{2})$ will be the angle whose cosine value is $\frac{-1}{2}$. We know this angle is $\frac{2\pi}{3}$ using the rules above.

Next, we determine that $\sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$. This is where things get a little more complicated. $\arctan(\frac{\sqrt{3}}{2})$ isn't an angle with reference angle $30°, 45°, 60°, 90°$. This is fine because, when we zoom out, we're asked to find $\sec(\arctan(\frac{\sqrt{3}}{2}))$. We know that the triangle formed by the angle $\arctan(\frac{\sqrt{3}}{2})$ has opp $= \sqrt{3}$ and adj $= 2$. We use the pythagorean theorem to find the hypotenuse:

$$(\sqrt{3})^2 + 2^2 = c^2$$
$$\implies c = \sqrt{3+4} = \sqrt{7}$$

Hence, hyp $= \sqrt{7}$. Even though the angle $\arctan(\frac{\sqrt{3}}{2})$ doesn't form a special triangle, we can still find its sec value using the sides of the triangle it forms:

$$\sec\left(\arctan\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{7}}{2}$$

Knowing this, $\sec\{\arctan[\sin(\arccos(\frac{-1}{2}))]\} = \frac{\sqrt{7}}{2}$.

4 Polynomial Long Division

Polynomial division isn't easily explained in notes, but it's very important!. Here is my recommended resource for polynomial division:

https://www.mathsisfun.com/algebra/polynomials-division-long.html

5 Remainder Theroem

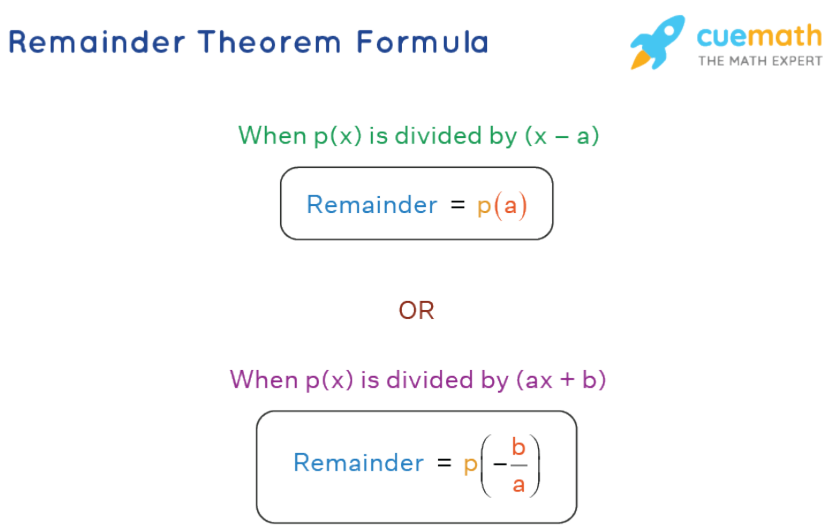

Figure 2: Credit: https://www.cuemath.com/algebra/remainder-theorem/

6 Descarte's Rule of Signs

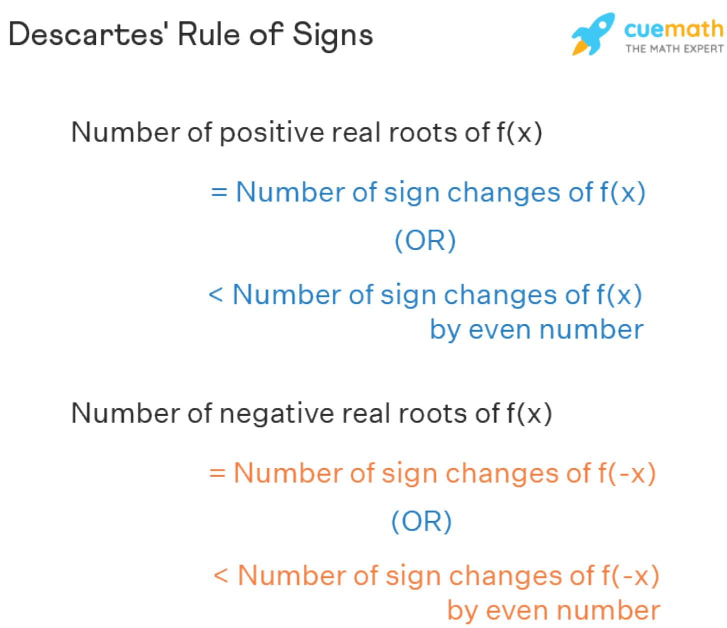

Figure 3: Credit: https://www.cuemath.com/algebra/descartes-rule-of-signs/

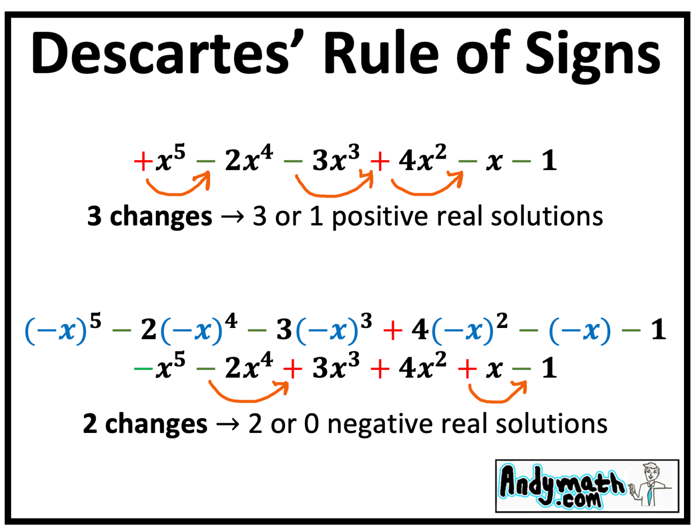

Figure 4: Credit: https://andymath.com/descartes-rule-of-signs/

7 Factors of Polynomials

If a polynomial has a zero of $x = a$ whose multiplicity is b, $(x - a)^b$ is a factor.

8 Included Angles

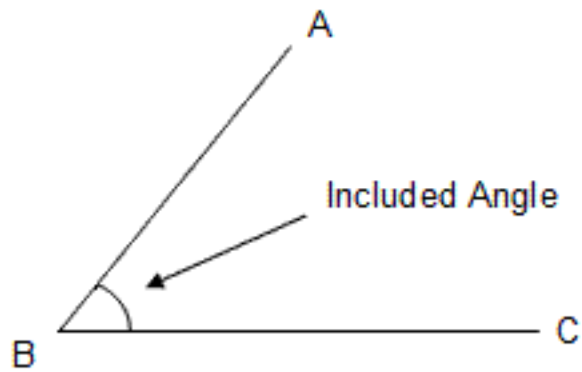

Figure 5: Credit: https://www.mathopenref.com/angleincluded.html

9 Law of Cosines

We use the law of cosines to "solve" (find all angles and sides) of a triangle when we are given either a) three sides or b) two sides and the included angle.

Consider the triangle

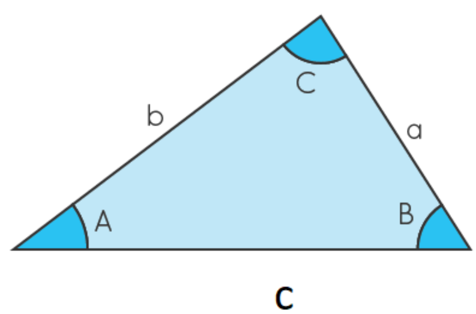

Figure 6: Credit: https://www.cuemath.com/questions/for-triangle-abc-with-sides-a-b-and-c-the-law-of-cosines-states-the-following/

with sides a, b, and c and angles A, B, and C. We define the law

$$a^2 = b^2 + c^2 - 2bc \cos A$$

which can be rewritten by swapping any two of the variables:

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

These can also be rewritten algebraically to solve for angles A, B, or C.

10 Law of Sines

The law of sines relates the proportions of sides to the sine values of the angles opposite the sides:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The rule is useful when we are given either a) two angles and one side, or b) two sides and a non-included angle.

11 Complex and Imaginary Numbers

Let z be the complex number $a + bi$. The conjugate of z is $\bar{z} = a - bi$. One useful property to remember is that $z\bar{z}$ is a real number.

12 Asymptotes and x-Intercepts of Rational Functions

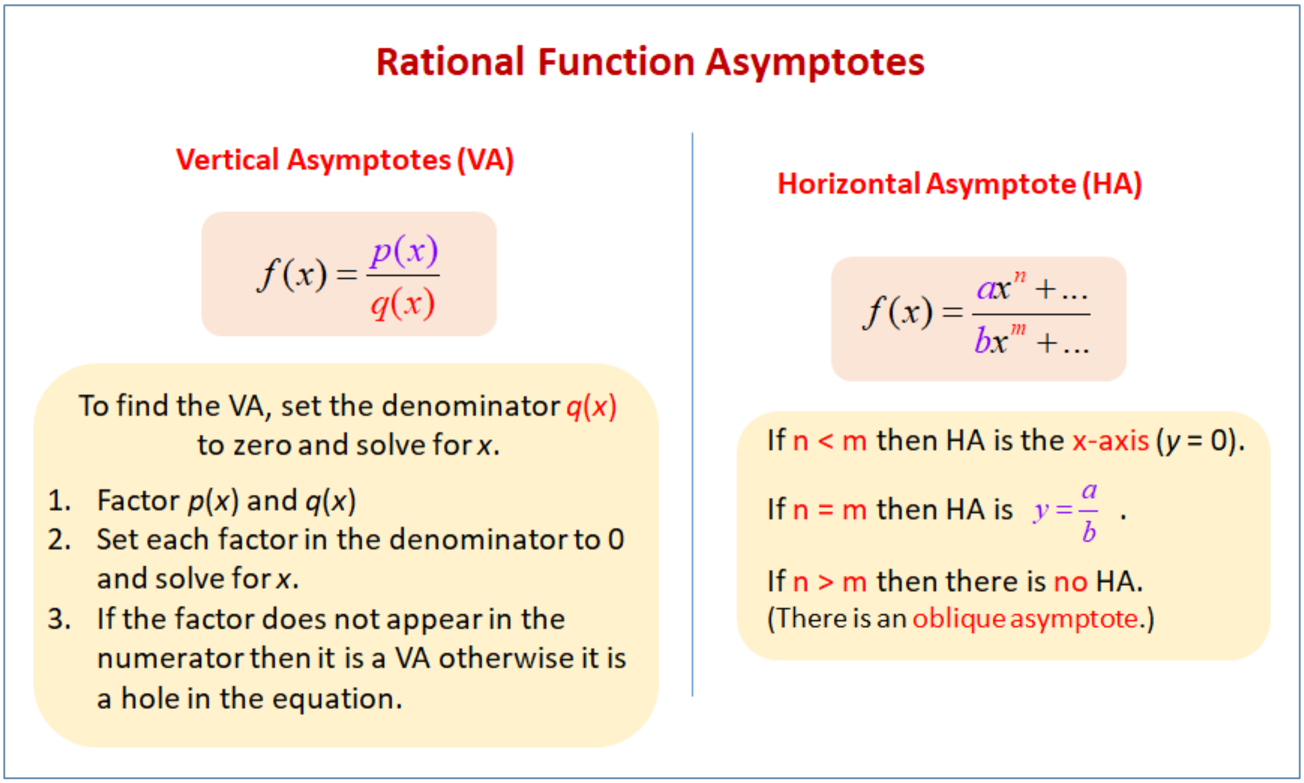

Figure 7: Credit: https://www.onlinemathlearning.com/rational-functions.html

The x-intercepts of a rational function are the values that make the numerator equal to zero.

If the degree of the top polynomial is one greater than the bottom polynomial, there exists an oblique or slant asymptote:

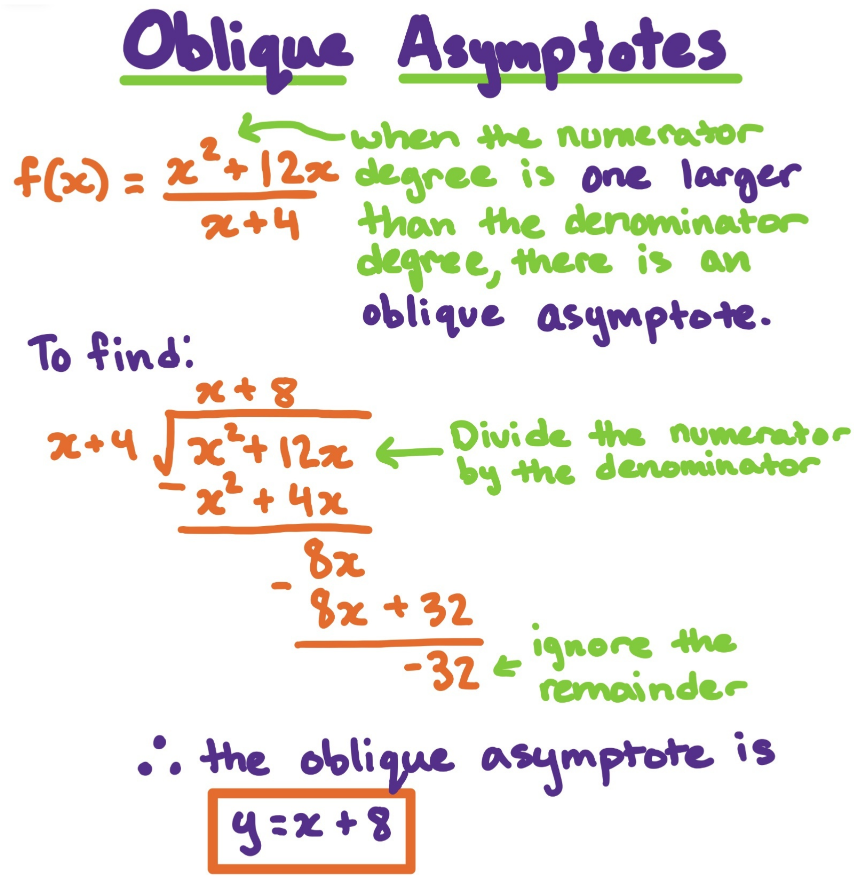

Figure 8: Credit: https://www.expii.com/t/oblique-asymptotes-of-rational-functions-5138

Some final notes about asymptotes:

Horizontal asymptotes and oblique asymptotes may be crossed by the rational function, but vertical asymptotes may not (because they're the values where the function is undefined).

To determine if some asymptote intersects a rational function, set them equal and look for a contradiction. If there isn't a contradiction, the rational function intercepts that asymptote at the value of x in that equality.

13 Compound Interest and the Natural Base

For an initial amount P, interest rate r, compounding rate per time period t, and time periods elapsed t, we define the final amount A as

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

If the compounding is continuous, meaning it's calculate infinitely many times in a given compounding period, the base is $e \approx 2.718$. Let's derive e:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{1 \cdot n}$$

Hence, to calculate the new amount given an initial amount P after t time periods with interest rate r, we use $A = Pe^{rt}$

For $P = 1000$, $t = 3$, and $r = 13\%$: $A = 1000e^{0.13 \cdot 3} \approx 1,476.98$

14 Logarithms

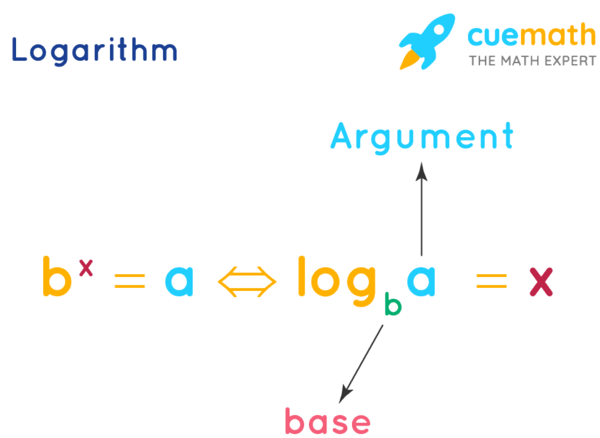

Figure 9: Credit: https://www.cuemath.com/log-formulas/

In plain language: the value of a logarithm is the power the base must be raised to in order to get the argument.

Logarithms are also the inverses of exponential functions. This means that for $f(x) = b^x$, $f^{-1}(x) = \log_b x$. Given this property, we know that a an exponential function and its corresponding inverse logarithmic function will reflect over the line $y = x$:

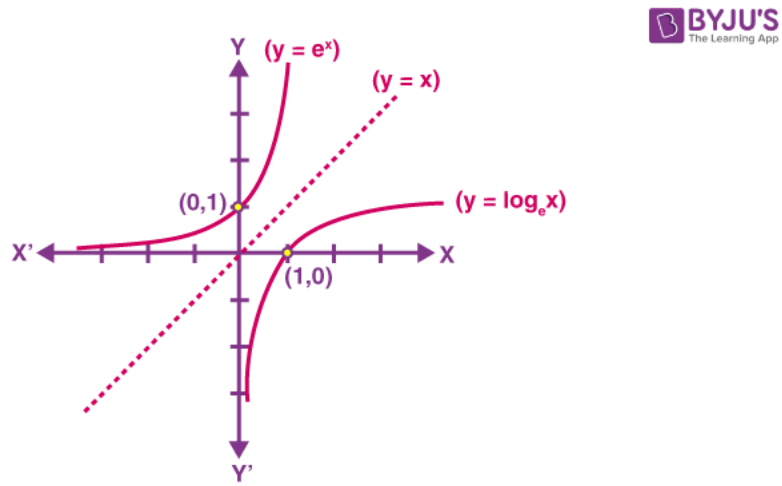

Figure 10: Credit: https://byjus.com/maths/exponential-and-logarithmic-functions/

Here are the rules to find the inverse of a logarithmic or exponential function:

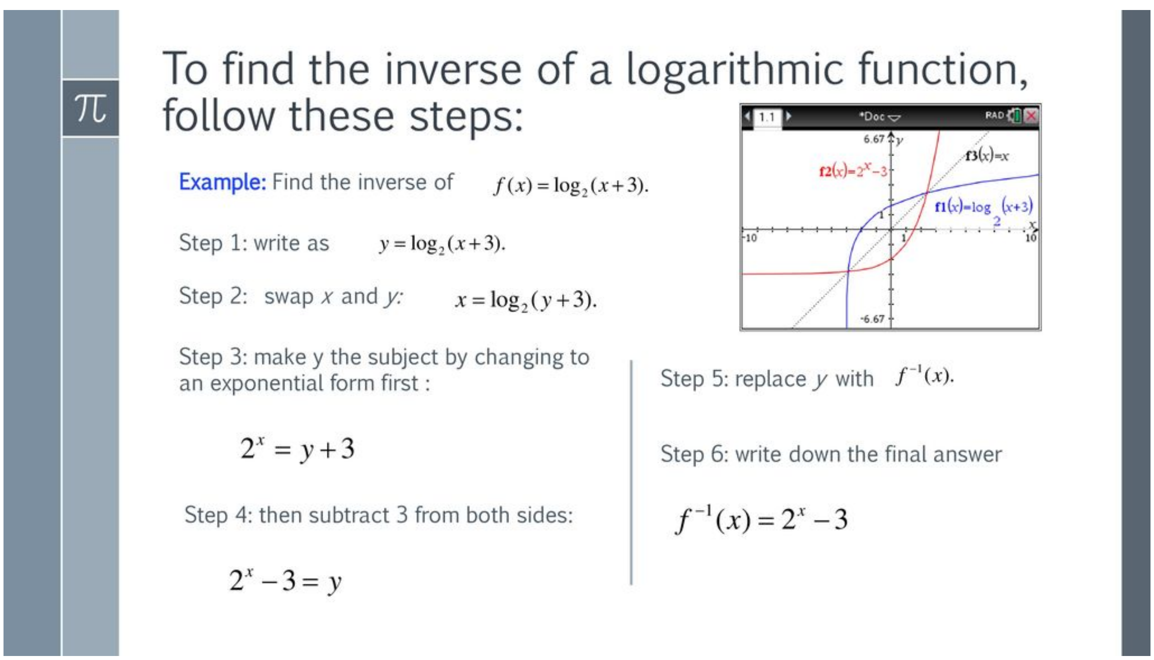

Figure 11: Credit: https://slideplayer.com/slide/12938368/

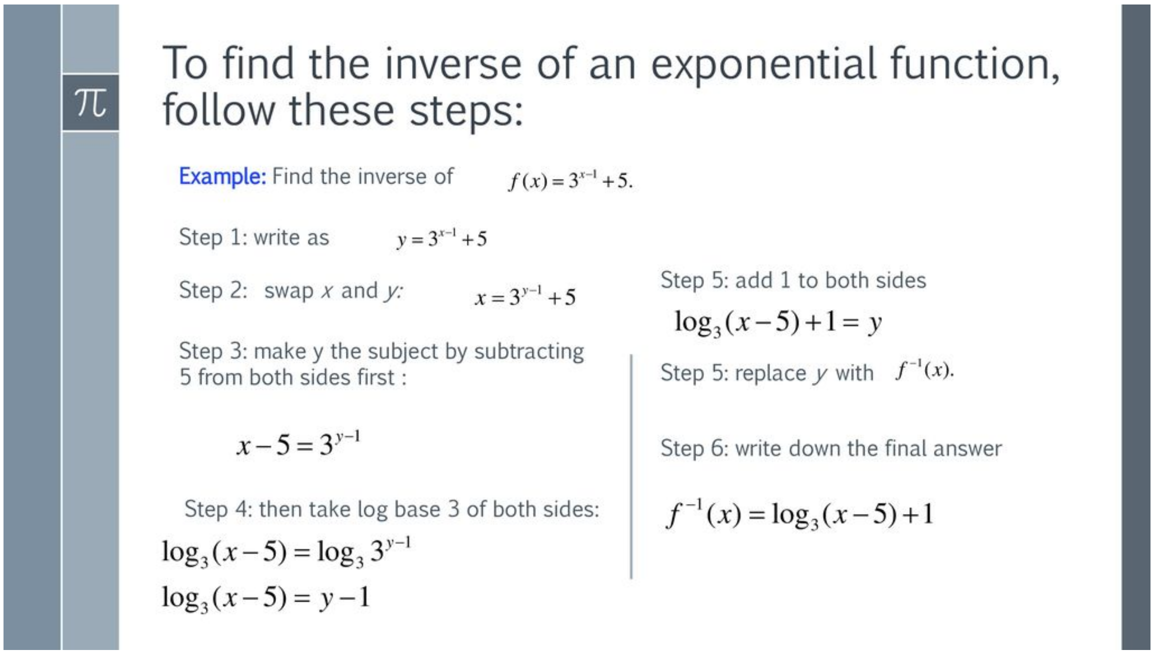

Figure 12: Credit: https://slideplayer.com/slide/12938368/

We didn't discuss logarithmic graphs in class, but here are some important properties to understand:

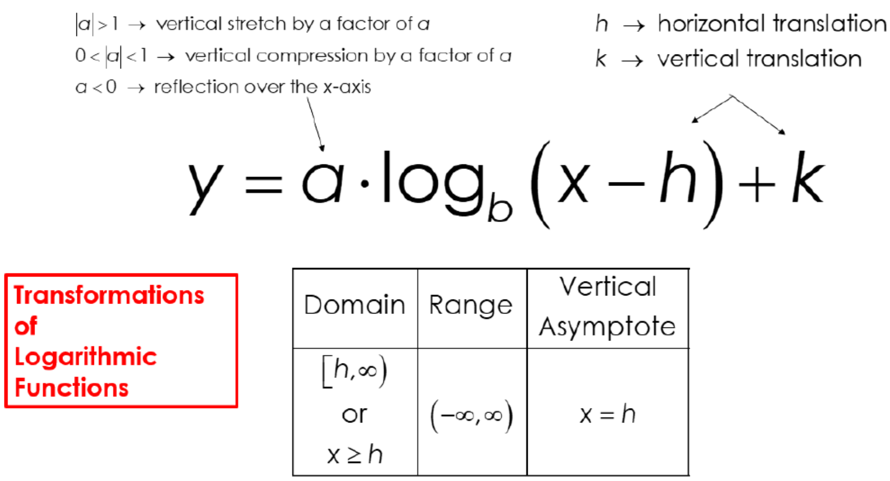

Figure 13: Credit: https://goodsifyet.shop/product_details/7113278.html

There exists an asymptote at $x = h$ because logarithmic functions may not have an argument less than 0. Similarly, they also may not have a negative base.

There are some rules for logarithms, too:

Rule of Logarithms

Rule Name	Property
Log of 1	$\log_b 1 = 0$
Log of the same number as base	$\log_b b = 1$
Product Rule	$\log_b(mn) = \log_b m + \log_b n$
Quotient Rule	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$
Power Rule	$\log_b m^n = n \log_b m$
Change of Base Rule	$\log_a b = \frac{\log_c b}{\log_c a}$ (OR) $\log_a b \,.\, \log_c a = \log_c b$
Equality Rule	$\log_b a = \log_b c \Rightarrow a = c$
Number Raised to Log	$b^{\log_b x} = x$
Other Rules	$\log_{b^n} a^m = \frac{m}{n} \log_b a$ $-\log_b a = \log_b \frac{1}{a}$ (OR) $= \log_{\frac{1}{b}} a$

Figure 14: Credit: https://www.cuemath.com/algebra/log-rules/

15 Fraction Exponents

$$x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

16 Exponential Growth and Decay with the Natural Base

Today's class only introduced the idea of exponential growth and decay, so I'll cover that here.

We define the general equation for exponential growth as

$$y(t) = ae^{kt}$$

where $y(t)$ is the value at time t, a is the initial value, and k is the growth constant.

Usually, we'll need to solve for k given information.

Problem: when will a population reach 50000 people if there are 10000 people initially and triples every two years?

Since the questions gives us the growth rate (”triples every two years”), we'll create the datapoint (2,30000) which repreesnts the population after two years.

We'll then plug this and given information into our equation $y(t) = ae^{kt}$ and solve for k, the growth constant:

$$\begin{aligned} 30000 &= 10000e^{2k} \\ \implies 3 &= e^{2k} \\ \implies \ln 3 &= \ln e^{2k} = 2k \\ \implies k &= \frac{\ln 3}{2} \end{aligned}$$

Now we can write the population as an exponential function of time:

$$y(t) = 10000e^{kt} = 10000e^{\frac{\ln 3}{2}t}$$

The question asks for the time when the population $y(t)$ is 50000, so we plug that in:

$$\begin{aligned} 50000 &= 10000e^{\frac{\ln 3}{2}t} \\ \implies 5 &= e^{\frac{\ln 3}{2}t} \\ \implies \ln 5 &= \ln e^{\frac{\ln 3}{2}t} = \frac{\ln 3}{2}t \\ \implies t &= \frac{\ln 5}{\left(\frac{\ln 3}{2}\right)} = \frac{2\ln 5}{\ln 3} \text{ years} \end{aligned}$$

Also, e is the base since it's easily differentiable in calculus. I had to look this up because I was confused why we didn't use simple bases (like $\frac{1}{2}$) for half-life decay.